The Very Early History of Trigonometry

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The early history of trigonometry, say for the time from Hipparchus through Ptolemy, is fairly well established, at least in broad outline (van Brummelen 2009). For these early astronomers plane trigonometry allowed the solution of an arbitrary right triangle, so that given either of the non-90° angles one could find the ratio of any two sides, or given a ratio of sides one could find all the angles. In addition the equivalent of the law of sines was known, although use infrequently, at least by Ptolemy. This skill was fully developed by the time Ptolemy wrote the *Almagest*, *ca* 150 CE (Toomer 1980), and he used it to solve a multitude of problems, some of them quite sophisticated, related to geometric models of astronomy. Ptolemy's sole tool for solving trigonometry problems was the chord: the length of the line that subtends an arc of arbitrary angle as seen from the center of a circle. Using a standard circle of radius 60, the *Almagest* gives a table of these chords for all angles between ½° and 180° in increments of ½°, and indeed Ptolemy gives a fairly detailed account of how one can compute such a table using the geometry theorems known in his time. Curiously, but not all that unusual for Ptolemy, it appears that some of the chord values in the *Almagest* were not in fact derived using the most powerful theorems that Ptolemy possessed (van Brummelen 1993, 46-73).

We also have evidence from Ptolemy that Hipparchus, working around 130 BCE, was able to solve similar trigonometry problems of about the same level of difficulty. For example, regarding finding the eccentricity and direction of apogee for the Sun's simple eccentric model, Ptolemy writes , Ptolemy writes in *Almagest* III 4:

> These problems have been solved by Hipparchus with great care. He assumes that the interval from spring equinox to summer solstice is 92½ days, and that the interval from summer solstice to autumn equinox is 92½ days, and then, with these observations as his sole data, shows that the line segment between the above-mentioned centres is approximately $1/24^{th}$ of the radius of the eccentre, and that the apogee is approximately 24½° in advance of the summer solstice.

The similar problem of finding the eccentricity and direction of apogee for the Moon's simple epicycle model is complicated by the moving lunar apogee. A glance at Figure 1 and a few moments consideration might give you some feel for the more advanced difficulty level of this particular problem. that Ptolemy explains in *Almagest* IV 6:

In this first part of our demonstrations we shall use the methods of establishing the theorem which Hipparchus, as we see, used before us. We, too, using three lunar eclipses, shall derive the maximum difference from the mean motion and the epoch of the [moon's position] at the apogee, on the assumption that only this [first] anomaly is taken into account, and that it is produced by the epicyclic hypothesis.

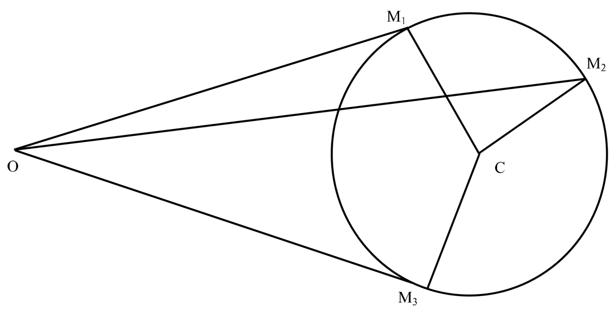

Figure 1. Consider a circle with center C and radius r. Let the distance $OC = R$. The angles M_1CM_2, M_2CM_3 and M_1OM_2, M_2OM_3 are given, and the problem is to find r/R. For a solution see *Almagest* IV 6 or Toomer 1973.

Finally, in *Almagest* IV 11 Ptolemy presents two trios of lunar eclipses that he says Hipparchus had used to determine the size of the first anomaly in lunar motion. Ptolemy gives just the results of Hipparchus' solutions, and from these we learn that while Hipparchus was certainly a capable user of trigonometry, he used a different set of numerical conventions than those used by Ptolemy. For example, while Ptolemy used a standard 360° degree circle with a radius of 60 parts, Hipparchus apparently specified the circumference of his circle as having 21,600 (= 360 x 60) parts, so that his diameter was about 6875 parts and his radius was about 3438 parts (Toomer 1973). We cannot, however, be sure whether Hipparchus used the same chord construct as Ptolemy, or perhaps just gave the ratio of side lengths corresponding to a set of angles. Nor can we be sure whether Hipparchus used a systematized table, or if he did, the angle increments of that table (Duke 2005).

One attempt to resolve these questions comes not from Greek or Roman sources, but from texts from ancient India that date from perhaps 400 – 600 CE. For many reasons, including the use of the circumference convention identical to that used by Hipparchus, and in spite of their appearance in India some six centuries after Hipparchus, it is has been proposed that these texts reflect a Greco-Roman tradition that is pre-Ptolemaic and largely

otherwise unknown to us (Neugebauer 1956, Pingree 1976,1978, van der Waerden 1961). These proposals have so far eluded definitive confirmation (and neither have any effective refutations appeared), but if they are true for the parts involving trigonometry, then it would seem plausible that Hipparchus' working set of tools included tables with 23 (non-trivial) entries of side ratios in angular increments of 3¾°, corresponding to chords in increments of 7½°, for we find exactly such tables in many Indian texts, always embedded in astronomical material that is extremely similar to early Greek astronomy.

We might be able to understand Hipparchus' use of trigonometry somewhat better if we had a little more idea how it was developed. There is a Greek source that might well be helpful in this regard, namely Archimedes' *Measurement of a Circle* (Heath 1897). Archimedes' mathematical methods in this paper are well-known: he uses the bounds

$$\tfrac{265}{153} < \sqrt{3} < \tfrac{1351}{780}$$

on $\sqrt{3}$ and then alternately circumscribes and inscribes a set of regular polygons around a circle, ultimately computing the ratio of the circumference of 96-sided polygons inside and outside the circle to the diameter of the circle, thus establishing bounds on π as

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}$$

What Archimedes actually computes in both cases (circumscribing and inscribing), however, are the ratios of the lengths of sides for a series of right triangles with smallest interior angle 30°, 15°, 7½°, 3¾°, (and partially $1\tfrac{7}{8}^\circ$), and so except for normalization many of the entries for the tables used in India and perhaps also by Hipparchus are computed in Archimedes' text, and all the entries are easily found using Archimedes' method.

Thus, denoting the opposite side, the adjacent side, and the hypotenuse by *a*, *b,* and *c* Archimedes finds for the circumscribed sequence of right triangles ratios of the following values:

	a	*b*	*c*
30°	153	265	306
15°	153	571	591 1/8
7½°	153	1162 1/8	1172 1/8
3¾°	153	2334 3/8	2339 3/8

The entries in the first row result from Archimedes' lower bound on $\sqrt{3}$, while the entries in row i+1 follow from those in row i using Archimedes' algorithm:

$$a_{i+1} = a_i$$
$$b_{i+1} = b_i + c_i$$
$$c_{i+1} = \sqrt{a_{i+1}^2 + b_{i+1}^2}$$

The ratios for the complementary angles 60°, 75°, 82½°, and 86¼° are trivially obtained by interchanging columns *a* and *b*, and we now have the ratios for eight of the 23 non-trivial angles in the sequence. We may get an additional eight values by applying Archimedes' algorithm to the angles 82 ½ °, yielding the table entries for 41¼° and 48¾°, to the angle 75°, yielding the entries for 37½°, 52½°, 18¾°, and 71¼°, and to the angle 52½ °, yielding the entries for 26¼° and 63¾°. Thus we get:

	a	*b*	*c*
41¼°	1162 1/8	1324 7/8	1762 3/8
37½°	571	744	937 7/8
18¾°	571	1682	1776 1/4
26¼°	744	1508 7/8	1682 3/8

and the ratios for the complementary angles again come from interchanging *a* and *b*.

Thus 16 of the 23 table entries are immediately available directly from Archimedes' text. To get the remaining seven entries it is necessary to repeat Archimedes' analysis beginning from a 45° right triangle and bounds on $\sqrt{2}$. If Archimedes used the bounds

$$\tfrac{1393}{985} < \sqrt{2} < \tfrac{577}{408}$$

then one would find for the sequence of circumscribed triangles ratios of the following values:

	a	*b*	*c*
45°	985	985	1393
22½°	985	2378	2573 7/8
11¼°	985	4951 7/8	5049
33¾°	2378	3558 6/8	4280 1/8

and the ratios for the complimentary angles 67½°, 78¾°, 56¼° again follow from interchanging *a* and *b*.

The analysis of the inscribed triangles follows the same algorithm but instead begins with the upper bounds on $\sqrt{3}$ and $\sqrt{2}$. The resulting bounds on the ratios are so close that for all practical purposes – let us remember, these are used for analysis of measured astronomical angles, and we use linear interpolation for untabulated angles – we can use either set, or their average, with no appreciable difference in results. Here is the entire set of entries:

	circumscribed		inscribed		circumscribed	inscribed
Angle	*a*	*c*	*a*	*c*	Base 3438	Base 3438
3 6/8	153	2339 3/8	780	11926	225	225
7 4/8	153	1172 1/8	780	5975 7/8	449	449
11 2/8	985	5049	408	2091 3/8	671	671
15	153	591 1/8	780	3013 6/8	890	890
18 6/8	571	1776 2/8	2911	9056 1/8	1105	1105
22 4/8	985	2573 7/8	408	1066 1/8	1316	1316
26 2/8	744	1682 3/8	3793 6/8	8577 3/8	1520	1520
30	153	306	780	1560	1719	1719
33 6/8	2378	4280 1/8	985	1773	1910	1910
37 4/8	571	937 7/8	2911	4781 7/8	2093	2093
41 2/8	1162 1/8	1762 3/8	5924 6/8	8985 6/8	2267	2267
45	985	1393	408	577	2431	2431
48 6/8	1324 7/8	1762 3/8	6755 7/8	8985 6/8	2584	2584
52 4/8	744	937 7/8	3793 6/8	4781 7/8	2727	2727
56 2/8	3558 6/8	4280 1/8	1474 1/8	1773	2858	2858
60	265	306	1351	1560	2977	2977
63 6/8	1508 7/8	1682 3/8	7692 7/8	8577 3/8	3083	3083
67 4/8	2378	2573 7/8	985	1066 1/8	3176	3176
71 2/8	1682	1776 1/8	8575 4/8	9056 1/8	3255	3255
75	571	591 1/8	2911	3013 6/8	3320	3320
78 6/8	4951 7/8	5049	2051 1/8	2091 3/8	3371	3371
82 4/8	1162 1/8	1172 1/8	5924 6/8	5975 7/8	3408	3408
86 2/8	2334 3/8	2339 3/8	11900 4/8	11926	3430	3430

In the table above, for each angle in col. 1 cols. 2–3 and cols. 4–5 give the lengths of the opposite side and the hypotenuse for the circumscribed and inscribed triangles, respectively, in Archimedes' method. Cols. 6 and 7 give the rounded length of the opposite side assuming the hypotenuse has length 3438 parts, corresponding to a circumference of 21,600 parts. Note that for all 23 angles the ratios for each angle are identical to the level of approximation used.

Therefore, we see that using Archimedes' method, and in many cases the very numbers that appear in his text, anyone could have assembled the table in increments of 3¾° that was used in India and might have been used by Hipparchus. The two steps needed to go beyond Archimedes are (a) a normalization convention, and (b) an interpolation scheme, and there seems no reason to doubt that any competent mathematician of the time would have the slightest trouble dealing with either issue. We are certainly in no position to say that Archimedes himself constructed the table, or who in the century between Archimedes and Hipparchus did it, but it is clear that by the time of Archimedes' paper all the needed tools and results were in place, except possibly for the motivation to actually organize the table.

We can, in fact, go even farther back into the very early history of trigonometry by considering Aristarchus' *On Sizes and Distances* (Heath 1913), and we shall see that a plausible case can be made that his paper could easily have been the inspiration for Archimedes' paper. The problem Aristarchus posed was to find the ratio of the distance of the Earth to the Moon to the distance of the Earth to the Sun. He solved this problem by assuming that when that the Moon is at quadrature, meaning it appears half-illuminated from Earth and so the angle Sun-Moon-Earth is 90°, the Sun-Moon elongation is 87°, and so the Earth-Moon elongation as seen from the Sun would be 3°. Thus his problem is solved if he can estimate the ratio of opposite side to hypotenuse for a right triangle with an angle of 3°, or simply what we call sin 3°. In addition, for other problems in the same paper Aristarchus also needed to estimate sin 1° and cos 1°.

Aristarchus proceeded to solve this problem is a way that is very similar to, but not as systematic as, the method used by Archimedes. By considering circumscribed (Fig. 2) and inscribed triangles (Fig 3) and assuming a bound on $\sqrt{2}$ Aristarchus effectively establishes bounds on sin 3° as

$$\tfrac{1}{20} < \sin 3^{\circ} < \tfrac{1}{18}$$

and, although he does not mention it, this also establishes bounds on π as

$$3 < \pi < 3\tfrac{1}{3}$$

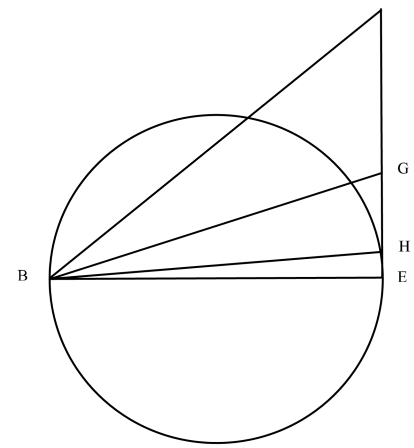

Figure 2. BE is a diameter of the circle, angle EBF is 45°, angle EBG is 22½°, and angle EBH is 3° (not to scale). Since EBG/EBH = 15/2 then GE/EH > 15/2. Since FG/GE = $\sqrt{2}$ > 7/5 then FE/EG > 12/5 = 36/15 and so FE/EH > (36/15)(15/2) = 18/1.

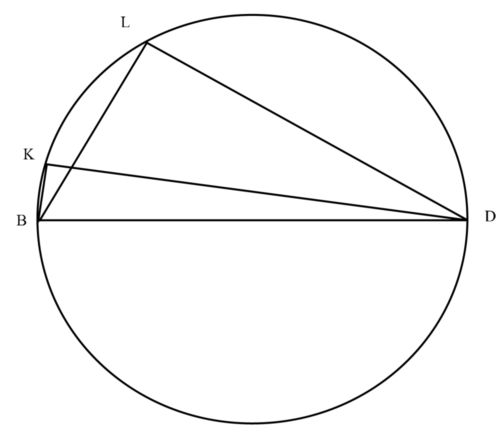

Figure 3. BD is a diameter of the circle, angle BDL = 30°, and angle BDK = 3° (not to scale). Since arc BL = 60° and arc BK = 6° then BL/BK < 10/1. Since BD = 2 BL then BD/BK < 20/1.

Later, in Propositions 11 and 12 Aristarchus proves using similar methods that

$$\tfrac{1}{60} < \sin 1^\circ < \tfrac{1}{45}$$

and

$$\tfrac{89}{90} < \cos 1^\circ < 1$$

always understanding, of course, that what we write as sine and cosine was to Aristarchus a ratio of sides in a right triangle. None of these bounds are particularly tight, and it is difficult to know if this was the best Aristarchus could do, or whether it was simply adequate for his purposes, which is apparently the case in any event.

The similarities between Aristarchus' and Archimedes' methods are clear: both assume bounds on a small irrational number, and hence effectively on the value of sin α for some relatively large angle, 60° or 45°, and through a sequence of circumscribed and inscribed triangles on a circle establish bounds on a target small angle, 3° for Aristarchus and $1\tfrac{7}{8}^\circ$ for Archimedes. Archimedes clearly realizes that this established bounds on π; Aristarchus may or may not have realized it, or might have not considered his bounds interesting enough to mention. Both Aristarchus and Archimedes are focused firmly on the relations between angles and ratios of sides in right triangles, neither ever using anything related to the chord construct used by Ptolemy. We know that Archimedes and Aristarchus exchanged correspondence, and we know that Archimedes was well aware of Aristarchus' work on the Earth–Moon–Sun distance problem. Indeed, Archimedes tells us that his own father also worked on the problem. In any case the parallels in the two calculations are quite striking, and it is not hard to imagine that Aristarchus' calculation could have been the inspiration behind Archimedes' calculation.

Coupled with the fact that the sin and not the chord is used also in the Indian texts, this suggests that the chord was introduced later rather than sooner, and certainly offers no encouragement to anyone claiming that Hipparchus used chords or that the sine was invented in India as an 'improvement' over the chord.

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An Early Use of the Chain Rule

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One of the most useful tools we learned when we were young is the chain rule of differential calculus: if $q(\alpha)$ is a function of α, and $\alpha(t)$ is a function of t, then the rate of change of q with respect to t is

$$\frac{dq}{dt} = \frac{dq}{d\alpha} \cdot \frac{d\alpha}{dt}$$

In the special case that $a(t)$ is linear in t, so $\alpha(t) = \alpha_0 + \omega_a (t - t_0)$, this becomes

$$\frac{dq}{dt} = \frac{dq}{d\alpha} \omega_a$$

If $q(\alpha)$ is a complicated function of α, for example

$$q(\alpha) = \tan^{-1}\left(\frac{-e \sin\alpha}{R + e\cos\alpha} \right)$$

then the computation of $dq/d\alpha$ is not necessarily easy. In this case

$$\frac{dq}{d\alpha} = \frac{-e/R\cos\alpha - (e/R)^2}{1 + 2e/R\cos\alpha + (e/R)^2}$$

so when e/R is small we have simply

$$\frac{dq}{d\alpha} \simeq -\frac{e}{R}\cos\alpha$$

In cases like this a practical alternative is to tabulate $q(\alpha)$ at small intervals $\Delta\alpha$ and then estimate $dq/d\alpha$ as a ratio of finite differences:

$$\frac{dq(\alpha)}{d\alpha} \simeq \frac{q(\alpha + \Delta\alpha) - q(\alpha)}{\Delta\alpha}$$

This particular function $q(\alpha)$ in our example is, of course, the equation of center for the simple eccentric (or, equivalently, epicycle) model used by Hipparchus and later Ptolemy for the Sun and the Moon (at syzygy), and it connects the mean longitude $\bar{\lambda}$ and true longitude λ according to

$$\lambda = \bar{\lambda} + q(\alpha)$$

where $\alpha = \bar{\lambda} - A$ and A is the longitude of apogee. As we shall see, Ptolemy very clearly knew that the rate of change with time of the true longitude λ is

$$\frac{d\lambda}{dt} = \omega_t + \omega_a \frac{dq}{d\alpha}$$

where ω_t and ω_a are the mean motion of the Moon in longitude and anomaly. Actually proving the chain rule is straightforward enough, but not entirely trivial, although perhaps in this simple case it might be guessed by dimensional analysis. As is often the case, Ptolemy gives no hint of how he came to know it.

It is, I think, not as widely appreciated as it might be that the result just given appears in Ptolemy's *Almagest*, not once but twice, and so was known at least as early the 2nd century CE, and very probably was known to Hipparchus in the 2nd century BCE, therefore nearly two millennia before the development of differential calculus (for standard treatments see, e.g. Neugebauer 1975, 122-124, 190-206 or Pedersen 1974, 225-226, 341-343).

The first occurrence of this result is found in *Almagest* VI 4. Ptolemy has just completed explaining how to compute the time $\bar{t}$ of some mean syzygy – a conjunction or opposition of the Sun and Moon in mean longitude – using their known mean motions and epoch positions in mean longitude and anomaly, and is ready to show how to estimate the time $t = \bar{t} + \delta t$ of the corresponding true syzygy. Therefore let us consider the case of a mean conjunction at some time $\bar{t}$, so that

$$\bar{\lambda}_S(\bar{t}) = \bar{\lambda}_M(\bar{t})$$

and work out what Ptolemy would do if he knew calculus.

Since we know the mean anomalies $\alpha_S(\bar{t})$ and $\alpha_M(\bar{t})$ at time $\bar{t}$ we can also compute the equations $q_S(\alpha(\bar{t}))$ and $q_M(\alpha(\bar{t}))$. At time t of true syzygy we have

$$\bar{\lambda}_S(t) + q_S(\alpha_S(t)) = \bar{\lambda}_M(t) + q_M(\alpha_M(t))$$

(with, of course, the addition of 180° on one side of the equation in the case of an opposition). Since the mean longitudes vary linearly in time we have simply

$$\bar{\lambda}_M(t) = \bar{\lambda}_M(\bar{t} + \delta t) = \bar{\lambda}_M(\bar{t}) + \omega_t \delta t$$
$$\bar{\lambda}_S(t) = \bar{\lambda}_S(\bar{t} + \delta t) = \bar{\lambda}_S(\bar{t}) + \omega_S \delta t$$

where ω_S is the mean motion of the Sun, so that

$$\bar{\lambda}_M(t) - \bar{\lambda}_S(t) = (\omega_t - \omega_S)\delta t = \eta \delta t = q_S(\alpha_S(t)) - q_M(\alpha_M(t))$$

Furthermore, since δt is small compared to the orbital period of the Moon, and even more so the Sun, we have

$$\begin{aligned} q_M(\alpha_M(t)) &= q_M(\alpha_M(\bar{t})) + \delta t \left.\frac{dq_M}{dt}\right|_{t=\bar{t}} + O(\delta t^2) \\ &\simeq q_M(\alpha_M(\bar{t})) + \omega_a \delta t \left.\frac{dq_M}{d\alpha}\right|_{t=\bar{t}} \\ q_S(\alpha_S(t)) &= q_S(\alpha_S(\bar{t})) + \delta t \left.\frac{dq_S}{dt}\right|_{t=\bar{t}} + O(\delta t^2) \\ &\simeq q_S(\alpha_S(\bar{t})) + \omega_S \delta t \left.\frac{dq_S}{d\alpha}\right|_{t=\bar{t}} \end{aligned}$$

noting that for the standard solar model of Hipparchus and Ptolemy the mean motions in longitude and anomaly of the Sun are equal since the solar apogee is tropically fixed.

Combining these and solving for δt gives

$$\delta t = \frac{q_S(\alpha_S(\bar{t})) - q_M(\alpha_M(\bar{t}))}{\eta + \omega_a \left.\frac{dq_M}{d\alpha_M}\right|_{t=\bar{t}} - \omega_S \left.\frac{dq_S}{d\alpha_S}\right|_{t=\bar{t}}}$$

Ptolemy, of course, does not know how to do a Taylor expansion approximation, but the result he gives is uncannily similar. First he instructs

us to estimate the true distance between the Sun and Moon at mean syzygy, which we see from the above is

$$q_S(\alpha_S(\bar{t})) - q_M(\alpha_M(\bar{t}))$$

He then says to multiply this by $^{13}\!/_{12}$ and to divide that result by the Moon's true speed, which he estimates as

$$0;32,56^{\circ/hr} - 0;32,40^{\circ/hr}\,(q(\alpha + 1^\circ) - q(\alpha))$$

where $0;13,56^{\circ/\mathrm{hr}}$ is the Moon's mean motion in longitude ω_t expressed in degrees per equinoctial hour, and similarly $0;32,40^{\circ/\mathrm{hr}}$ is the hourly mean motion in anomaly. Note also that

$$q(\alpha + 1^\circ) - q(\alpha) = \frac{q(\alpha + 1^\circ) - q(\alpha)}{1^\circ} = \left.\frac{\Delta q}{\Delta \alpha}\right|_{t=\bar{t}}$$

so Ptolemy has estimated $dq/d\alpha$ with a finite difference approximation, and furthermore chosen an interval $\Delta\alpha = 1°$ that, at first sight, cleverly avoids an otherwise bothersome division operation.

So in the end his estimate of the correction δt to the mean time $\bar{t}$ is, in units of equinoctial hours,

$$\delta t = \frac{q_S(\alpha_S(\bar{t})) - q_M(\alpha_M(\bar{t}))}{\frac{12}{13}\left(0;32,56^\circ + 0;32,40^\circ \left.\frac{dq_M}{d\alpha_M}\right|_{t=\bar{t}}\right)}$$

which compares very closely to the more exact result derived above, the only differences being that he has two approximations in the denominator: first, he gives

$$\frac{12}{13} \times 0;32,56 = 0;30,24$$

which is a good approximation to η = 0;30,8, and second he neglects the term proportional to $dq_S/d\alpha_S$ which is smaller than the already small (compared to 0;32,56) derivative of the Moon's anomalistic equation of center.

Although Ptolemy's scheme of estimating $dq/d\alpha \simeq q(\alpha + 1^\circ) - q(\alpha)$ is certainly one option, it is not necessarily the best option when the task is to make the estimate using a table of $q(\alpha)$ values, especially the table found in the *Almagest*, where the table entries are either 3° or 6° apart.. For one reason, it requires two table interpolations. Yet these can be easily avoided if the instructions are instead to find the interval in which α lies, i.e. find α_i and α_{i+1} such that $\alpha_i \le \alpha < \alpha_{i+1}$ (which can be done by inspection), and then estimate $dq/d\alpha$ using

$$\frac{dq(\alpha)}{d\alpha} = \frac{q(\alpha_{i+1}) - q(\alpha_i)}{\alpha_{i+1} - \alpha_i}$$

which, given the piecewise linearity of the table, is about the best estimate you can make in any case without resorting to a higher order interpolations scheme. Furthermore, the quotients on the right hand side of the above equation could all be precomputed and included in the table and would be useful for all table interpolations, but that is not done in the *Almagest*. Thus, the procedure that Ptolemy describes would make a lot more sense, especially in terms of computational efficiency, if the table was compiled with an interval of 1° in the variable α. Strabo tells us that for geography Hipparchus did compile length of the longest day at intervals of 1° in terrestrial latitude, so it would not be too surprising if Hipparchus had 1° tables for lunar, and for that matter, solar anomaly.

Ptolemy goes on to estimate how close to the nodes the Moon has to be before an eclipse is even possible. For lunar eclipses this is straightforward, but for solar eclipses a rather involved calculation involving lunar parallax is required, lunar parallax having already been analyzed in detail in *Almagest* V 17–19. Ptolemy then discusses the allowed intervals (in months) between lunar and solar eclipses. Besides the common six month interval, it turns out that lunar eclipses can also occur at five month, but not seven month, intervals, and solar eclipses can occur at not only both five and seven month intervals, but also at one month intervals, provided the observers are at widely different locations, including being in different (north and south) hemispheres.

Related to all this is a passage in Pliny's *Natural History*, written *ca*. 70 CE, which says

> It was discovered two hundred years ago, by the sagacity of Hipparchus, that the moon is sometimes eclipsed after an interval of five months, and the sun after an interval of seven; also, that he becomes invisible, while above the horizon, twice in every thirty days, but that this is seen in different places at different times.

For Hipparchus to know all this, and in particular the part about solar eclipses at one month intervals, requires that he had a significant amount of computational skill, including a reasonable command of lunar parallax. Indeed, Ptolemy tells us that Hipparchus wrote two books on parallax. Therefore it is hardly a stretch to presume, with Neugebauer 1975, 129 and Pedersen 1974, 204, that Hipparchus already knew the eclipse material reported by Ptolemy in the *Almagest*, including the use of the chain rule discussed above.

Besides using the instantaneous speed to estimate the time difference between mean and true syzygy, it is also needed to estimate for lunar eclipses the time difference between first and last contact with the Earth's shadow, and in the case of total lunar eclipses, the time interval of complete immersion (and, of course, similarly for solar eclipses).

The second occurrence of the use of the chain rule is in *Almagest* VII 2 concerning retrograde motion. Ptolemy begins by recalling Apollonius' treatment (from perhaps 180 BCE) of the simple epicycle model, in which the distance from the Earth to the epicycle center is constant. The ratio of a particular pair of geometric distances is, according to Apollonius' theorem, equal to the ratio of the speed ω_t of the epicycle center to the speed ω_a of the planet on the epicycle, both of which are constant in the simple model. However, in the case of the more complicated *Almagest* planetary models – the equant for Saturn, Jupiter, Mars, and Venus and the crank mechanism for Mercury – the relevant ratio is between the true speeds v_t and v_a as observed from Earth, which are not constant, and this once again involves using the chain rule, just as above:

$$\frac{d\lambda_t/dt}{d\lambda_a/dt} = \frac{\omega_t + \frac{dq}{dt}}{\omega_a + \frac{dq}{dt}} = \frac{\omega_t + \omega_t' \frac{dq}{d\alpha}}{\omega_a + \omega_t' \frac{dq}{d\alpha}}$$

where ω_t' is ω_t diminished by $1^{\circ/cy}$ to account for the sidereally fixed apogees in the *Almagest* planetary models. In this case Ptolemy does not actually explain how to compute the numerical derivatives for $dq/d\alpha$, but the numerical values he gives for each planet confirm that he was using the tables of mean anomaly in *Almagest* XI 11, or something pretty close to them.

Returning now to eclipses, the natural question to wonder about is whether this careful estimate of the instantaneous speed is worth the effort? For example, how much difference would it make in eclipse predictions if in the calculations the mean speed η was used instead of the accurately calculated speed? In order to investigate this questions I have computed, using the *Almagest* rules, all 977 lunar eclipses from –746 to –130.

The speed is used two ways. First, it is used to compute the difference in time between mean and true conjunction, the eclipse being taken to occur at true conjunction rather than at minimum distance from the shadow center. This latter approximation is a good one, the time difference between true conjunction and minimum distance averaging less than 2 minutes and never exceeding 6 minutes, no matter which speed, mean or instantaneous, is used. On the other hand, the estimates of the actual time of true conjunction vary by about 19 minutes on average, and for about 40% of lunar eclipses the time difference exceeds 20 minutes, with a maximum difference of about 48 minutes.

Second, the speed is used to compute the duration of partial and total eclipse. Considering just partial eclipses, which are probably the easiest to time and show the largest effect in any event, the average difference in computed duration is about 12 minutes. and for about 14% of lunar eclipses the difference of computed duration of partial eclipse time interval exceeds 20 minutes, with a maximum difference of about 41 minutes. The differences that exceed 20 minutes arise when the eclipses have low magnitude, so that a relatively small change in the latitude of the Moon can result in a relatively large change in the path length needed to cross the shadow.

Altogether then, it seems reasonable to me that these differences in predicted absolute time and duration of lunar eclipses, while not exactly dramatic, are large enough to suggest a motivation for the ancient astronomer to compute the times using the instantaneous rather than the mean speed.

All of this by no means implies that differential calculus as we know it was understood by ancient mathematicians, but it does show that when they needed to solve a special problem, such as the one above, they were in some cases able to do it.

References

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