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Research note

Extremal tides: rigorous computation for the many-body case

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Summary. A rigorous method is developed for the exact solution of the extrema of the total tidal field on a spherical celestial body, disturbed by a multiplicity of gravitating point masses distributed in three dimensions at distances large relative to the disturbed body's size. A short program is provided for convenient use of the method. As an illustration, maximal solar tides due to planetary attraction are calculated for the Solar System 1964–1991.

Introduction – tidal investigations

Recent decades have seen an abundance of attempts to correlate tidal effects on the Earth, Sun or stars with:

(a) volcanic eruption (Mauk & Johnston 1973; Heaton 1975);
(b) earthquakes (Tamrazyan 1967; Gribben & Plagemann 1976);
(c) Earth-spin (Gribben & Plagemann 1976, chapters 7 and 9);
(d) solar flares (Blizard 1968);
(e) sunspots (Loomis 1866, p. 244; Bigg 1967; Takahashi 1968; Wood 1972) (such a study is proceeding currently at one of the major national observatories (private correspondence));
(f) even starspots (Mullan 1974).

With one exception, I will not enter here into the question of validity regarding any of the alleged correlations. My primary concern is with the fact that a number of these investigations would be rendered more efficient and more accurate by the availability of a rigorous means of determining the height (which we will henceforth call the upper-case Tide) – and the location on the disturbed body – of the highest point of the total equilibrium tide due to the sum tidal forces of any given moment's configuration of disturbing bodies.

The need for such a method of finding the Tide is dramatized by the expedients (resorted to in its lack) of some of the above-cited papers, e.g. Takahashi (1967), which evidently calculates planet-induced tides around the entire solar equator to find the maximum (the Tide) by trial, or Wood (1972), which employs the primitive and ultimately quite misleading device of basing its deductions upon the sum of tidal components only along one pre-chosen planet's radius-vector.

Therefore, I have devised a completely general method of solution, which will determine the magnitudes and locations of the extrema of the total tide on a spherical body disturbed by any number of point masses, distributed in any way in three dimensions (at distances which are large relative to the size of the disturbed body).

Such a method is likely to find uses not yet tried. For example, we are just entering into an era of close-up investigation of other planetary systems (most possessing more than one satellite). And, to add a speculative possibility: it may be that 'irregular' variable stars' pulsations will be explained in part from tidal effects of companions.

Calculating the total tidal field

For a complete solution to the problem, one starts with the well-known equation for the tidal potential U due to point mass m (at distance r' from the disturbed body's centre):

$$U = -Gmr'^{-3}r^2 P_2(\cos\theta) \tag{1}$$

where G = universal gravitation constant; $\mathbf{r}$ = the field point; $\mathbf{r}'$ = source point; θ = the angle between these two vectors; P_2 is the second Legendre polynomial. (It is presumed that $r \ll r'$ throughout, so that higher orders are not required here.)

Equation (1) is most conveniently dealt with in a Cartesian form (employing the usual repeated-index summation-convention below, except where indicated):

$$U = -(\alpha/2)\,(3x_h x'_h x_k x'_k - R^2 x_q x_q) \tag{2}$$

where $\alpha = GmR^{-5}$ and $R = r'$.

The grad of equation (2) provides an expression for the tidal gravitational field $\mathbf{g}$ (directly due to the external mass m):

$$g_j = -\partial U/\partial x_j = \alpha(3x_k x'_k x'_j - R^2 x_j) \tag{3}$$

which, for multiple tidal-influencing bodies m_i, becomes (summing i once only, over all m_i):

$$g_j = \alpha_i\,(3x_k x'_{ki} x'_{ji} - R_i^2 x_j) \tag{4}$$

(where $\alpha_i = Gm_i R_i^{-5}$), the total tidal force. (The word 'directly' is used because the tidal shift itself produces a change (quantified in the general second order tidal problem via the Love numbers h_2 and k_2) due to the additional effects of: (a) radial displacement of field point on the disturbed surface (h_2); (b) self-gravity of the tidally shifted matter (k_2). The tidal height is affected by the factor $1 + k_2$; radial field, by the factor $1 + h_2 - 1.5\,k_2$.) If one sets

$$C_{jk} = \alpha_i(3x'_{ji}x'_{ki} - R_i^2\,\delta_{jk}), \tag{5}$$

a symmetric tensor, then equation (4) reduces to a simple expression

$$g_j = C_{jk}x_k. \tag{6}$$

Equations (4)–(6) provide an exact solution for $\mathbf{g}$ as a function of position $\mathbf{r}$.

Finding extrema on the disturbed sphere

Determination of the extremal tidal field strengths on the surface of a disturbed spherical body (e.g. the Sun) now requires the diagonalization of C.

The secular equation will be cubic, since the problem is in three dimensions. The three eigenvectors will represent the axes of symmetry of the ellipsoidal equipotential surface resulting from the total tidal disturbance. After equation (6) is transformed to the eigenvector reference frame, C is diagonal; thus each eigenvalue λ_n corresponds to a simple solution (n not summed):

$$g_n = \lambda_n x_n \tag{7}$$

radially along the eigenvector axis x_n.

At the sphere's surface, $x_n = a$, the radius (effectively constant), so each

$$g_n = a\lambda_n ; \tag{8}$$

thus (merely adjusting for the constant factor a) the eigenvalues λ_n are identical to the extremal values (maximum, saddlepoint, minimum) for the tidal strength g. This suggests a valid short cut for one interested only in the maximum (the Tide), i.e. merely solve C's secular equation. (The rest of the diagonalization process is only concerned with orientation.) The largest of the three λ_n is the Tide.

The method described above may easily be generalized to the case of tidal disturbance by a continuum of matter.

The tidal height

To determine, in length units, the equilibrium height h (above the surface of the hypothetical undisturbed sphere) of the tidal distortion, use the equipotential condition: the change of gravitational potential of matter shifting radial distance h in response to the tidal field must balance the tidal potential U; that is (taking M for the disturbed mass):

$$\int_a^{a+h} (-GMr^{-2})\, dr + U = 0. \tag{9}$$

Since $h \ll a$, equation (9) becomes

$$h = -[a^2/(GM)]\, U. \tag{10}$$

(Thus, if the height, radial field, and potential are normalized, we have $h = g_r = -U$ everywhere on the sphere, although to be scrupulous, we must say that, on the actual tidally distorted non-spherical (ellipsoidal) surface, the radial displacement (h) of course causes a potential change nullifying U (from the equipotential condition). For the Sun, this same radial displacement (h) doubles g_r since the Sun's $k_2 \doteq 0$ and (because a fluid body's $h_2 = 1 + k_2$) $h_2 \doteq 1$.)

Substituting equation (1) into equation (10) yields, for one disturber (using $r = a$ on the disturbed sphere, and $r' = R$), the simple relation:

$$h = (m/M)\,(a^4/R^3)\, P_2(\cos\theta). \tag{11}$$

For the single-disturber case, the Tide obviously occurs at $\theta = 0$ ($P_2 = 1$), where we set $h = H$; thus, from equation (11):

$$H = a\,(a/R)^3\,(m/M). \tag{12}$$

As a useful illustration, we find, on the Sun S, the Tide H_S induced by an adopted ET unit disturbance, namely, that due to the mean distance Earth E. Substituting into equation (12) the mass ratio (Earth + Moon)/Sun = 1/328900 and $a_S/R_E = 16'01''.18 = 1/214.5954$ and $R_E = 1$ AU $= 1.496 \times 10^{11}$ m, we find the ET unit tidal height.

$$H_S = 0.2145 \text{ mm}. \tag{13}$$

However, it must be noted that the foregoing treatment works so simply for the Sun only because most of the Sun's mass is concentrated relatively near its centre (because of central mass concentration, the Sun's $k_2 \doteq 0$).

Sample computation – solar Tide 1964–1991

As an example of the method's use, the planet-induced Tide on the Sun was computed (all nine planets – elliptical orbits) every 5 day throughout a period of 9725 day (26.63 yr) from Julian Day 2438520.5 to JD 2448245.5 (Besselian dates 1964.34–1990.97). The solar Tide's highs and lows over time are listed in Table 1, expressed in ET units (equation 13). To find the Tide in millimetres, simply multiply the Tide values of Table 1 by the ET unit height H_S, 0.2145 mm.

Peaks in the solar Tide exceeding 7 ET units occur in 1977.97, 1984.47 and 1990.02. The highest (at 1990.02) is about 1.6 mm.

With regard to the alleged 'Jupiter effect' (Gribben & Plagemann 1976) – a solar Tide-induced super 1982 earthquake – we note from Table 1 that 1982 has the second *lowest* Tide

Table 1. Temporal relative maxima and minima of solar Tide (ET units), 1964.34–1990.97.

Date	Tide	Date	Tide	Date	Tide	Date	Tide	Date	Tide	Date	Tide	Date	Tide
1964		.30	5.7	.91	3.2	.93	2.0	.80	1.8	.31	1.9	.25	2.2
.43	4.5	.34	2.9	.95	4.8	1976		.85	2.6	.50	6.7	.34	5.3
.47	5.7	.37	3.8	1972		.02	5.5	.86	2.3	.67	2.4	.37	3.6
.54	1.8	.43	2.6	.00	1.7	.06	3.2	.92	5.2	.74	5.1	.41	5.5
.71	6.4	.53	4.9	.10	4.1	.10	4.3	1980		.78	3.6	.55	1.9
.75	3.8	.57	3.9	.14	3.5	.20	1.6	.07	3.0	.85	5.2	.57	2.7
.78	4.2	.61	6.0	.19	5.5	.26	5.0	.09	3.2	.96	2.1	.60	2.4
.88	2.3	.74	2.3	.27	1.7	.29	3.5	.11	2.7	.98	3.9	.70	5.5
.95	5.7	.78	3.0	.43	7.0	.36	6.0	.17	4.5	1984		.78	5.0
.99	3.8	.80	2.4	.49	4.9	.48	3.4	.20	4.0	.03	1.8	.83	6.0
1965		.85	4.3	.51	5.0	.52	4.5	.34	5.3	.15	4.5	.88	2.4
.04	5.0	.98	3.0	.64	2.8	.56	3.0	.44	2.0	.20	2.7	.90	2.8
.16	2.6	1969		.67	3.5	.75	6.1	.58	5.6	.23	4.3	.94	2.7
.19	3.2	.04	3.6	.70	2.2	.80	2.9	.62	3.3	.29	2.4	1988	
.21	2.1	.06	3.4	.81	4.6	.84	3.2	.65	5.4	.41	4.5	.07	5.9
.33	5.7	.08	3.5	.88	2.4	.88	3.0	.76	1.6	.42	4.4	.11	3.2
.40	4.5	.11	3.3	.92	3.9	1977		.82	3.7	.47	7.2	.14	3.6
.44	6.0	.27	6.5	.95	2.0	.00	6.5	.85	2.5	.67	2.3	.22	2.0
.51	2.0	.31	3.6	1973		.04	4.2	.90	4.9	.71	4.7	.31	6.3
.68	5.9	.34	4.0	.08	5.5	.08	4.5	1981		.75	3.2	.34	4.6
.84	2.0	.43	2.2	.13	4.2	.18	2.8	.03	1.6	.83	4.1	.39	6.1
.92	6.3	.50	4.4	.17	6.2	.24	4.6	.07	2.4	.93	2.3	.53	2.0
.97	4.7	.54	2.8	.28	1.9	.26	3.2	.08	2.2	.96	4.0	.56	3.3
1966		.58	4.8	.40	6.5	.36	5.8	.14	4.8	1985		.59	2.8
.03	5.6	.70	1.7	.61	2.4	.45	3.4	.16	4.7	.00	2.7	.70	4.7
.16	2.4	.75	3.8	.65	4.4	.49	4.8	.32	6.0	.12	5.8	.76	4.0
.17	2.8	.77	2.6	.69	3.4	.54	1.8	.37	2.5	.18	3.8	.80	5.5
.19	2.5	.84	5.3	.81	4.9	.64	4.1	.38	2.6	.21	5.1	.85	2.1
.32	4.8	.97	3.3	.87	2.3	.68	3.5	.44	1.7	.29	2.4	1989	
.38	3.2	1970		.90	2.9	.73	5.6	.55	5.2	.38	4.1	.05	6.7
.41	5.1	.01	4.4	.93	2.1	.81	1.8	.59	2.9	.41	3.8	.20	1.9
.48	1.8	.07	3.0	1974		.97	7.1	.63	4.0	.45	6.5	.28	6.0
.59	4.2	.24	6.4	.06	6.2	1978		.72	1.6	.59	3.0	.31	4.3
.61	3.8	.30	3.4	.10	4.8	.03	4.9	.78	5.0	.60	3.1	.38	5.8
.66	6.4	.31	3.7	.14	6.6	.04	5.0	.83	3.6	.65	2.2	.50	2.2
.81	1.8	.40	1.7	.26	1.9	.18	3.2	.85	5.5	.70	5.2	.53	3.9
.89	5.8	.48	5.3	.31	3.0	.20	3.6	1982		.75	4.2	.57	2.0
.94	4.4	.52	3.2	.33	2.2	.24	2.4	.03	1.9	.79	4.3	.67	4.6
1967		.57	4.1	.38	5.4	.35	4.9	.18	4.6	.96	2.6	.75	3.0
.02	5.1	.66	2.3	.52	3.2	.43	2.4	.23	4.4	1986		.78	4.6
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.15	3.4	.75	2.8	.58	2.4	.51	2.0	.36	2.2	.15	4.4	.94	4.8
.19	1.7	.83	5.8	.62	4.8	.62	5.1	.38	2.5	.19	5.5	.97	4.7
.31	4.6	.94	3.7	.66	4.5	.67	3.8	.40	2.4	.28	1.8	1990	
.36	2.5	.98	4.9	.80	6.3	.70	5.8	.53	5.7	.35	4.0	.02	7.4
.39	3.9	1971		.86	2.9	.81	2.1	.57	3.0	.38	3.1	.20	2.2
.44	2.1	.04	2.4	.88	3.3	.94	6.6	.60	3.3	.42	5.4	.26	5.1
.56	4.9	.21	5.7	.93	2.2	1979		.68	2.0	.55	2.1	.30	3.6
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.79	2.2	.34	2.5	.08	3.9	.23	2.8	.86	5.8	.67	5.8	.50	4.0
.87	4.7	.46	6.4	.12	5.8	.36	4.4	1983		.80	5.0	.54	2.6
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.09	2.5	.69	3.9	.36	5.5	.60	5.9	.15	4.0	.09	6.1	.82	2.6
.12	3.6	.72	2.5	.51	2.6	.64	4.3	.22	3.3	.13	3.8	.91	4.4
.16	2.9	.81	5.3	.78	6.9	.68	6.1	.26	5.0	.16	4.8	.94	4.0

peaks of the 27 years (1964–1990) covered in Table 1. (The weakest year of all is 1980, the time of the latest sunspot cycle's peak, which the 'Jupiter effect' had correlated to a supposed solar Tide relative maximum.) The curious circumstance is directly due to the fact that the authors' Sun Tide predictions were based on Wood's method – discussed above – which included the tidal effects of only Venus, Earth and Jupiter using *circular* orbits. Ironically, the inclusion of Jupiter's non-trivial orbital eccentricity inverts the 'Jupiter effect', virtually reversing the high and low peak-Tide times.

It is hoped that the availability of a reliable table of solar Tide behaviour for 2 2/3 decades, and a short program for computation (Appendix B), will ease the labour (and subsequent evaluation) of future researches in this area, ensuring a firm calculational basis from the start.

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Appendix A: proof of extremeness

If the potential, equation (2), is partially differentiated and set zero (to find the stationary points), in tandem with the equation

$$x_p x_p = a^2 \qquad \text{(A1)}$$

for the surface of the disturbed sphere as a constraint associated with a Lagrange multiplier λ, the result is:

$$\partial U/\partial x_j = \lambda \partial/\partial x_j (x_p x_p) = \lambda 2 x_p \delta_{pj} = 2\lambda x_j. \qquad \text{(A2)}$$

Substituting equations (2)–(6) into equation (A2), and altering the arbitrary constant λ by the factor, –2:

$$g_j = C_{jk} x_k = \lambda x_j. \qquad \text{(A3)}$$

A comparison of equation (A3) to equations (6)–(7) establishes the effective identity of the eigenvalue and Lagrange multiplier, and the fact that the potential U is indeed stationary on the three eigenvector principal axes, $\mathbf{x}_n$.

Since g_r (the radial component of $\mathbf{g}$) is everywhere on the sphere equal to $-2U/a$ (one may remove the constant ratio, $-2/a$, via normalization), the radial component of the tidal field is also stationary on these axes (where, incidentally, $\mathbf{g}$ is entirely radial).

To prove that the magnitude of the full vector g is stationary there, as well, we repeat the above test, now upon g^2, again using equation (A1) as constraint, this time with a Lagrange multiplier λ', and substituting equation (6):

$$\partial/\partial x_j(g_k g_k) = 2g_k\, \partial g_k/\partial x_j = 2C_{km}x_m\, \partial/\partial x_j(C_{ki}x_i) = 2\lambda' x_j. \tag{A4}$$

This reduces to (taking unnecessary advantage of C's symmetry):

$$C_{jk}C_{km}x_m = D_{jm}x_m = \lambda' x_j; \tag{A5}$$

to show that the former eigenvector solutions satisfy this new condition, put equation (A3) into equation (A5), twice successively:

$$C_{jk}(\lambda x_k) = \lambda(C_{jk}x_k) = \lambda(\lambda x_j) = \lambda^2 x_j = \lambda' x_j. \tag{A6}$$

Thus, letting $\lambda' = \lambda^2$, the condition equation (A5) for stationary g (or g^2) is indeed met at the principal axes.

As to whether the eigenvector positions represent maxima, minima or saddlepoints: if the second partial derivatives of $U(= -\tfrac{1}{2}C_{jk}x_jx_k)$ and $g^2(= D_{jk}x_jx_k)$ are examined in the eigenvector reference frame (i.e. after diagonalization, when $U = -\tfrac{1}{2}\lambda_j x_j^2$, $g^2 = \lambda_j^2 x_j^2$; j summed once), under the constraint, equation (A1); then, at the nth stationary point, the Hessian of constrained U is:

$$(\lambda_n - \lambda_j)\,\delta_{jk}; \tag{A7}$$

while that of constrained g^2 is:

$$2(\lambda_j^2 - \lambda_n^2)\,\delta_{jk}; \tag{A8}$$

j is not summed in equations (A7) and (A8).

In the general case of non-equal eigenvalues, we set the convention:

$$\lambda_1 > \lambda_2 > \lambda_3. \tag{A9}$$

Noting from equation (5) the nullity of C's trace, we have (since the trace is invariant under the similarity transformation which diagonalizes C):

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{A10}$$

From equations (A9) and (A10), we see that $\lambda_1 > 0$ and $\lambda_3 < 0$, and that:

$$\lambda_1^2 > \lambda_2^2 < \lambda_3^2. \tag{A11}$$

Applying inequality (A9) to equation (A7) and inequality (A11) to equation (A8), it is easily seen that: at eigenvector #1, U is a minimum; at #2, g is a minimum; at #3, U is a maximum; g is a maximum at #1 if $\lambda_1^2 > \lambda_3^2$; at #3 if the reverse. Generally, for multi-body planet or satellite systems,

$$\lambda_1^2 > \lambda_3^2 > \lambda_2^2. \tag{A12}$$

For the Solar System in particular: eigenvector #1 is nearly parallel to the ecliptic; the same is usually true of #2; #3 is usually about perpendicular to the ecliptic (exceptions) being Sun-planets near-line-ups: near-degenerate λ_2 and λ_3, high λ_1).

Appendix B: program for automatic calculation of tidal extrema

From the foregoing, the following BASIC-PLUS program has been compiled. Given the disturbed body's mass and radius and the masses and positions of the disturbing bodies, the program computes the tidal extrema (length units) and associated principal axes of the disturbed body's equilibrium tidal ellipsoid.

```
9000 MAT C=ZER:MAT Q=ZER:MAT R=ZER
9010 B=A↑4/M
9020 FOR I=1 TO S
9030 R(I)=R(I)+L(J,I)↑2 FOR J=1 TO 3
9040 R(I)=SQR(R(I))
9050 W(I)=B*M(I)/(2*R(I)↑3)
9060 V(I)=3*W(I)/(R(I)↑2)
9070 FOR J=1 TO 3
9080 FOR K=1 TO 3
9090 C(J,K)=C(J,K)+V(I)*L(J,I)*L(K,I)
9100 C(J,K)=C(J,K)-W(I) IF J=K
9110 NEXT K
9120 NEXT J
9130 NEXT I
9140 R=C(2,3)*C(1,1)*C(3,2)+C(1,3)*C(2,2)*C(3,1)+C(1,2)*C(3,3)*C(2,1)
      -C(1,3)*C(3,2)*C(2,1)-C(3,1)*C(1,2)*C(2,3)-C(1,1)*C(2,2)*C(3,3)
9150 Q=C(1,1)*C(2,2)+C(2,2)*C(3,3)+C(3,3)*C(1,1)
      -C(1,2)*C(2,1)-C(2,3)*C(3,2)-C(3,1)*C(1,3)
9160 Q=SQR(ABS(Q)/3)
9170 V=-R/(2*Q↑3)
9180 W=SQR(1-V↑2)
9190 P=ATN(W/V)
9200 P=P+PI IF V<Ø
9210 FOR N=1 TO 3
9220 F(N)=2*Q*COS((P+(N+1)*2*PI)/3)
9230 D=(C(2,2)-F(N))*(C(3,3)-F(N))-C(2,3)*C(3,2)
9240 N2=C(2,1)*(C(3,3)-F(N))-C(3,1)*C(2,3)
9250 N3=C(3,1)*(C(2,2)-F(N))-C(2,1)*C(3,2)
9260 P(1,N)=1
9270 P(2,N)=-N2/D
9280 P(3,N)=-N3/D
9290 Q(N)=Q(N)+P(J,N)↑2 FOR J=1 TO 3
9300 X(J,N)=P(J,N)/SQR(Q(N)) FOR J=1 TO 3
9310 NEXT N
```

Key:

A	= radius of disturbed body.
M	= mass of disturbed body.
S	= total number of disturbing bodies.
$M(I)$	= mass of Ith disturbing body.
$L(J,I)$	= position vector of Ith disturbing body.
$F(N)$	= Nth extremal tidal height, in length units (see discussion of equations 3, 4, and 13).
$X(J,N)$	= normalized Nth axis of symmetry of tidal ellipsoid; multiply by A for position where tidal height = $F(N)$.