Ancient Solstices

Ancient Solstice-Determiners’ Delicate Voyage ’Twixt Random Error’s Scylla and Systematic’s Charybdis

Tihon Finds Hipparchos’ – 157/6/26 18th Solstice
Its Significance and Neat Surprise-Solution

New Light on Hipparchos’ Calendar,
Solar Elements, & Year-Length

A  Summary & Unwelcome Shock-Confirmation of DIO Prescience

In 2010, Anne Tihon (www.springer.com/us/book/9789048127870) meticulously analysed a recently recognized papyrus (P.Fouad 267A, fortunately recommended to her expert examination by Jean-Luc Fournet) bearing: Hipparchos’ – 157/6/26 Summer Solstice, use of his hitherto-unknown 500s solar longitude tables (one of them Kallippic), a new precession rate for the tropical points, also a new ancient yearlength which D.Duke soon correctly reasoned was based on comparison to Meton’s – 431 S.Solstice. Below, we show how ancient solstices were determined outdoors — as well as detailing the problems Hellenistic scientists had to balance, to achieve an accurate estimate of a solstice’s hour. We also examine why the best ancient scientists preferred solstices to equinoxes as bedrocks for their calendars; and we consider the newly-available – 157 solstice’s implications for dating some of Hipparchos’ astronomy. Curiously, no commentator on the papyrus’ – 157 solstice has yet remarked that the 1st and only prior paper to propose (§K) Hipparchos sought a – 157 solstice & used Kallippic mean solar motion is Rawlins 1991W. Do non-citers believe DIO happened only by blind luck to improbably [a] hit upon the now-papyrus-confirmed date of Hipparchos’ 1st try at a solstice (& orbit), [b] induce his Kallippic solar speed?!

B  Journal for the History of Astronomy Biggies’ 4 Solstice Adventures

Sending History-of-Astron’s MacArthur Genius Up to 9th Grade

The laugh-crying need for a competent article on mathematical and historical matters regarding ancient solstices may perhaps be brought home to the reader by a swift foray here into the wisdom on the subject that’s been emanating from academé’s two most highly-placed and expensive Experts1 in the field of ancient astronomy. [It would not be necessary to highlight the weird stuff that follows here, except that — typically for cohesive, wagon-circling cults — despite years of opportunity (and DIO nudges), the perps have not retracted on-the-record a single one of the strange-science adventures we enumerate below.]

B1 A.Jones, sometime Princetitutee, now at NYU’s hugely endowed Inst. for the Study of the Ancient World, Boardmember-for-Life at history of astronomy’s “premier” (Schaefer 2002 p.40) Journal for the History of Astronomy (& JHA’s discoverer of the Winter Equinox: Jones 1991H p.119), has added to JHA’s rep (www.dioi.org/jha.htm#kqlz) for meticulous refereeing by rejecting in its pages the reliable standard ancient method (Al-majest 1.12) for finding latitude&obliquity via solstices, using equinoxes instead: Jones 2002E, a paper taken rather too seriously by PU’s history of early trig, van Brummelen 2009, p.65 n.76, though with fair citation of DIO 4.2 p.56’s (or Rawlins 2009S p.20’s) stark Table 1. (Unlike Jones, who persists in nonciting this Diller-DR table’s perfect data-fit, to fake Jones 2002E’s viability, not even producing his own table! Do not miss fn 10 below.)

1Despite their here-appreciated screwball gags, each of our roastees has made solid contributions to knowledge, as seen at, e.g., §B4, DIO 4.3 §13 §D8, DIO 11.2 cover [owed to Duke&Jones]. DIO 12 §2.
J.Evans 1998 p.206, spent decades misunderstanding the 9th-grade-level method used by
ancestors to measure solstices, an achievement recognized by DR at R.Newton 1991 fn 20:
One of the more amusing moments in [HamSwerdlow 1981], which RRN is
too polite to note, is [HamSwerdlow 1981’s] sarcastic mock astonishment
while commenting upon a key RRN discrimination: “most remarkable of all,
that solstices could be observed with more accuracy than equinoxes.” That
RRN is correct (in the very judgement which HS attack as “remarkable” folly) is obvious to any unprejudiced scientist familiar with the instrumental
problems involved. (See the lucid discussion at R.Newton 1977 pp.81-82 or [G1 here].) So, oft-mentioned ancient astronomical-observer-cardinalists (excluding
[indoor] Ptolemy) ... depended primarily upon solstices for gauging the year’s length: Meton, Euklemon, Kallippos, Aristarchos, Hipparchos. (Hipparchos observed numerous equinoxes [30]; but even his year-lengths were
based upon solstices: see, e.g., [Rawlins 1991H] eq.8 [& below eqs.32&34].) However, Swerdlow, an historian [then] with the official rank of professor [at
U.Chicago’s Astron.Dep’t] cannot understand this elementary point: during a
gloriously delirious passage (p.527) in his prominent 1979 attack on Newton
(in American Scholar [Phi Beta Kappa] 48:523 ...). Swerdlow argues:
At the time of the solstice, the meridian altitude of the sun changes
by less than fourteen seconds of arc per day, and measuring this
quantity, let alone any fraction of it, was obviously ridiculous.
The only ridiculous aspect of this astounding piece of reasoning is that a
member of the University of Chicago’s Dep’t of Astronomy should so con
spicuously exhibit his touching innocence of the implications of 1-year
calculus and of the standard technique known as “equal altitudes”. It is easy to
see that Hist.sci archon Swerdlow’s reasoning is essentially equivalent to
insisting that the time a vertically oscillating body reaches maximum altitude
cannot be determined since at that moment it lacks vertical motion!

B3 Far from admitting his elementary misunderstanding, invincibly-ineducable Swerd
low keeps promoting the same reasoning’s validity a decade later, in the very Journal
for the History of Astronomy paper (Swerdlow 1989 p.36) which got him his MacArthur!

C Precisely Determined Ancient Solstices
C1 Given the hist.astron center’s continuing problems in the area of solstices (e.g.,
Rawlins 2009S §F3), it will help if we cite (and later list: Table 3) what we have hitherto
possessed of outdoor ancient solstices where the hour not merely the date is known. After
discounting those (Table 1) truncated to day-epoch — Meton’s (~431) & Aristarchos’
(~279) and the faked solst (Table 2) of Ptolemy (~140) — we find that we have just
four so far, most only by modern reconstruction, not direct attestation. (The exception is
~146, confirmed by P.Fouda 267A: §M4.) The ~329 S.Solstice launching Kallippos’
upcoming new stellar-year is reconstructable by realizing (§J4, Rawlins 1985H) that his pioneering
yearlength (nearly 3 centuries before Julius Caesar’s Sosigenes), \( Y_k = 365^2/4 \) arose from
his comparison of his own S.Solstice observation to Meton’s famous Athens ~431/6/27
S.Solstice, which was typically (for calendarists) truncated to the beginning (sunset for
Athens) of the 2nd period containing the event. So add 1023Y or 37255/12 to the start of
Meton’s calendar to find the solstitial moment of the Kallippos calendar’s start:
\[ -431/6/27 3/4 + 102 \cdot \left(365^2/4\right) = -329/6/28 1/4 \] (1)

(+3^b error). Like logic allows reconstruction of Aristarchos’ ~279 solstitial observation,

we may also use this method to estimate the Solstices’ date. (Note: ‘solst’ is set as a day
even if a yearlength is computed by Rawlins 1985H [& Rawlins 1991H eq.8].) In any case,
starting with Babylonian Astronomical Cuneiform Text (BM55555), whose Greek-based year-length is
\[ \gamma_{Y_{M}} = 365^2/4 \cdot 44'51'' = 365^2/73/297, \] (3)
we know from Rawlins 1991H that Hipparchos’ 135BC solstice (which he used to find
his final “UH” solar orbit: [ibid §C] occurred 297 of ACT 210’s years \( Y_{U_{M}} \) or 108478^d [§P6

\[ 0.2200 \times 108478 = 238851.6 \] (4)}
below) after Meton’s Solstice as misunderstood by Hipparchos. (Who interpreted Meton’s start-of-day as dawn instead of Athenian sunset. Perhaps just to find or force a fit to the overlong Metonic lunisolar scheme? See below at §4P5 & §Q1.) Rawlins 1991H eq.6:

\[
-431/6/27 1/4 + 297 \cdot (365^{1/2}/73/297) = -134/6/26 1/4
\]  

(4)

[Meaning pair-means for several d can ensure reliably accurate solst.
]

D Hellenistic Astronomers’ Outdoor Empiricism

The three accurate solstices cited (eqs.1-2&4) add to the accumulated evidence that Greek astronomers were anything but the dreamy, data-inventing crtiters that certain truly dreamy historians imagine. See, e.g., our comments (at Rawlins 2008R §A) on Mufa god-pop Neugebauer’s strange vision. Other evidences of Greek empiricism’s accuracy & prirmacy (Rawlins 2008Q §K4 & n.9) include the half-percent precision of Greeks’ basis-measure for finding the Earth’s radius (§1B3) — and more spectacularly their three lunar periods (§3 fn 27; www.dioi.org/thr.htm), each accurate to better than 1 part in a million.

E Truncated Solstices

We list all extant day-start-truncated solstices (§§C1-C2&E1-E3) in Table 1.

E1 Meton’s calendar started on \(-431/6/27 3/4\) since Athens’ day began at sunset. As late as a century after, Kallippos knew the original Meton calendar epoch and (eq.1) founded his year-length upon it — though (§C1&P4) Hipparchos later misconstrued Meton’s S.Solstice by \(-12^4\), making its error \(-29^6\), which caused (along with eq.31) huge systematic errors in later astronomers’ yearlength estimates (§Q1; Rawlins 1999 §B6).

F Equal-Alitudes: How the Ancients Determined Solstices

F1 As noted at §B2, DIO has for decades asserted (against Muffia-MacArthur geniusum) that ancient solstices were observed via Equal-Alitudes. Understanding the method shouldn’t challenge a high-schooler.

F2 Starting \(d\) days before the Solstice, as the Sun transits (culminates) at Local Apparent Noon (LAN), the observing astronomer records in degrees and arcminutes the altitude \(h\) of the Sun’s center (preferably just a few degrees below the LAN culmination). This noon will be called \(t_1\). By obvious symmetry, the LAN Sun’s altitude will be back near \(h\) at \(d\) days after Solstice, LAN-culminating at a time which will be called \(t_2\). The midpoint between the two times is then taken as the Solstice-hour \(t_{\text{MidPt}}\):

\[
t_{\text{MidPt}} = \frac{t_1 + t_2}{2}
\]  

(5)

[Obviously, the two times’ relation to \(d\) is (see further at §§G3&§J1-J2 & eqs.19-21):

\[
d = \frac{t_2 - t_1}{2}
\]  

(6)

G Solstice-Observation Technique: Going Beyond Naïve Eq.5

G1 The great accuracy-advantage of solstices vs equinoxes is this: if there is uncertainty in adopted solar parallax, atmospheric refraction, the transit-instrument’s mounting or secular settling or arc-ruuling-uniformity, then an equinox-timing is corrupted (§B4) by each’s systematic error. But not a solstice, since all these errors’ effects on \(t_1\) & \(t_2\) are nearly the same while of opposite sign, thus leaving eq.5 unaffected. Yet solstitial determination has its own problems, which are [a] lesser, but [b] serious and (except for random-error problems, which are smaller with equinoxes) completely different from the traditional bothers for equinox-observations.

4The ultimate new proof, that Hellenistic scientists had adopted Babylon’s sexagesimal measure for angles as early as the 3rd century BC, is found here at §J: Archimedes’ masked solar diameter brackets.

5Astronomers will see that we are merely using \(h\) to measure solar declination \(\delta\), in order to find two times on either side of Solstice when \(\delta\) is the same. Generally, finding the exact time \(t_2\) when the 2nd estimate of \(\delta\) exactly matches (that which occurred at \(t_1\)) will require interpolation — since only by rare luck does the post-Solstice \(t_2\) match the earlier noon \(t_1\) almost exactly at noon.

6Our present annual version of the technique has a diurnal parallel (fn 2) often used by pre-GPS-era explorers. (Among others: the Isaac Hayes & Rob’s Peary expeditions.) Secondary-school classes teach an analogous method for finding when a thrown ball reaches maximum height: §B2.
If the Sun’s motion were uniform, it is obvious from symmetry that LAN solar altitudes \( h_1 \) & \( h_2 \) measured at respective times \( t_1 \) & \( t_2 \) with an instrument set at constant solar altitude \( h \) so that

\[
h_2 = h_1
\]

would ensure that the corresponding solar longitudes are each at the same angular distance \( S \) from the S.Solst point:

\[
90^\circ - \phi_1 = \phi_2 - 90^\circ = S
\]

Assuming symmetry, the average of the two times of eq.5 would be exactly the sought quantity: the time \( t_{SS} \) of the S.Solst.

What very slightly but aggravatingly upsets the ideal eq.5 situation is the Earth’s elliptical orbit. The vital elements were in \(-157\) and thereabouts:

\[
\text{Apogee } A = 66^\circ.1 \quad \text{eccentricity } e = 0.0176
\]

The non-uniform solar motion entailed by the asymmetry\(^7\) of the Sun’s elliptic motion causes a \textit{systematic error} that becomes quadratically \textit{larger} (eq.13), the larger the number of days \( d \) on either side of the Solstice one chooses to take observations at — even while the process’ \textit{random} error becomes \textit{smaller} for greater \( d \) (eq.18). So picking the ideal \( d \) is a delicate choice (§3), whose pitfalls we now examine. [Note: Many equations to follow here are approximations — though marked as equalities if the roughness is slight.]

### H Charybdis

#### H1 An equation for the asymmetry-caused longitudinal systematic error \( q \), of an Equal-Altitudes-obtained \( t_{SS} \), may not have been previously published; so we have derived (and have substituted eq.9 values into) the following simple formula for \( q \) as a function of \( S \), the number of longitude-degrees on either side of the solstice one chooses to start & finish at:

\[
q = \frac{\pi e \cos A}{3} S^2 = -0.0075S^2
\]

with \( q \) in arc-minutes and (again) \( S \) in degrees.

#### H2 We all know the Sun moves about \( 1^\circ/day \), so obviously \( S \) is nearly equal to \( d \) — near enough for the difference to be largely ignorable here. Nonetheless, we supply useful approximations, relating \( q \) to the asymmetry-caused solstice-error \( H \) in hours, for solar motion near a Summer Solstice during Hipparchos’ era,

\[
H = \frac{24 \cdot 365^2 \cdot 2425}{60 \cdot 360^3} q/(1 - 2 e \sin A) = 0.42q
\]

and relating \( S \) to \( d \):

\[
d = \frac{365^2 \cdot 2425}{360^3} S/(1 - 2 e \sin A) = 1.05S
\]

Combining eqs.10-12 yields our ultimate desired simple practical formula (valid for the range of \( d \) that knowledgeable ancients would wish to use) expressing systematic error in hours \( H \) as a function of the Equal-Altitudes symmetric (ere&af) interval \( d \) in days:

\[
H = -0.0075 \cdot 0.42 \frac{d^2}{1.05^2} = -0.0029d^2
\]

where the minus-sign reflects that for \(-157\) the error of the Equal-Altitudes Method will cause naively (eq.5) deduced \( t_{SS} \) to be too-early by \( H \) hours.

[Analysis simplified at www.dioi.org/cs/pdf, §E2.]

\(^7\)Things were easy in 1245AD, when the solar apogee arrived at longitude \( 90^\circ \). Had this obtained in Hipparchos’ era, our entire discussion of asymmetry here would be superfluous.

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**I Scylla**

We now turn from systematic error to \textit{random error}.

If solar altitude \( h \) could be measured perfectly, the foregoing Charybdis section would be a complete error analysis. But the measure of \( S \) is from visual determination of altitude \( h \), which can be measured to no better than \( 1/10000 \) of a radian (Rawlins 2002B eq.1), called here the Optimal standard-deviation \( \sigma_{opt} \) for human vision — and contrasted with widely-assumed Ordinary visual discrimination (oft apt in-practice: §B4), \( \sigma_{Ord} \equiv 1^\circ \).

#### Eq.14 causes an uncertainty (\( \sigma_{SS} \)) in an Equal-Altitudes-determined S.Solst time \( t_{SS} \) which requires statistical evaluation. So, to find an accurate \( t_{SS} \), we initially need to know how strongly \( h \)-uncertainty produces uncertainty in hours of solar motion.

Since LAN solar motion and the solar declination \( \delta \) virtually differ by a constant, we start by gauging the statistical relation of longitude \( \phi \)’s uncertainty \( \sigma_\phi \) to \( h \)’s uncertainty \( \sigma_h \):

\[
\frac{\sigma_h}{\sigma_\phi} = \frac{\Delta h}{\Delta \phi} \equiv \frac{\Delta \delta}{\Delta \phi} = \frac{\tan \epsilon \sin S}{\pi \cdot \tan \epsilon S}
\]

(15)

\( (\sigma_\phi, \sigma_h \) in arcm, where obliquity \( \epsilon \) was \( 23^\circ.7 \) in Hipparchos’ era. Also (a statistical parallel to eq.11), we find the effect of \( \sigma_\phi \) upon observed \( t_{SS} \)’s uncertainty \( \sigma_{SS} \) (in hours):

\[
\frac{\sigma_{SS}}{\sigma_\phi} = 0.42/\sqrt{2}
\]

(16)

where the \( \sqrt{2} \) reflects \( t_{SS} \)’s dependence (eq.5) upon not one but two \( h \) measures, averaged.

Combining eqs.15, 16, & 12 establishes standard-deviation ratios:

\[
\frac{\sigma_{SS}}{\sigma_h} = \frac{0.42}{\sqrt{2}} \frac{S}{\pi \sqrt{2} \tan \epsilon S} = 0.42 S \frac{\sigma_h}{S} \frac{39}{41} \sigma_h = 41 \frac{d}{d} \sigma_h
\]

(17)

We note that when \( S = 0 \) (the Swerdlow-Moment: §B2), uncertainty \( \sigma_{SS} \) in an Equal-Altitude-Method-obtained S.Solst-time is infinite — as it obviously should be.

To evaluate \( t_{SS} \)’s uncertainty \( \sigma_{SS} \) as a function of \( d \) for Optimal and Ordinary visual discrimination, we exploit eq.17 by substituting into it eq.14’s respective values for \( \sigma_h \):

\[
\text{Optimal } \sigma_{opt} = 0.42 \frac{d}{d} \sigma_h \quad \text{Ordinary } \sigma_{Ord} = \frac{d}{d} \sigma_h
\]

(18)

The above considerations show that accuracy to well within the ancient-quoted (Almagest 3.1) allowance of \( 6^\circ \) error was possible, so it should be no surprise that all three of the firm outdoor solstices of Table 3 are accurate within the uncertainty-estimates of the present section: after all, for accurate data correctly rounded to \( 6^\circ \) precision, the implicit error-range is \( \pm 3^\circ \).

### J Balance

We next weigh the tricky choice an ancient solstice-observer had to face. If chosen \( d \) is too small, he is prey to the quirky Scylla of corruption of his project by random error of indeterminate size and even sign. But if the ancient astronomer over-counters that danger by opting for too-large \( d \), he leans too near Charybdis’ traverse swirl and thus intolerable systematic negative error. (Hartner 1977 & Thurston 2001 cite pre-telescopic observers’ \( d \) ranging from \( 45^\circ \) [eq.13: \(-1^\circ/4 \text{ syst.error}) to \( 8^\circ [1^\circ/4 \text{ random error for Ordinary eq.18, 2^\circ for Optimal})]. To estimate an ideal Balanced interval \( d_{BAL} \), we can combine eqs.13&18 to ensure that \( H \) and \( \sigma_{SS} \) are about the same size:

\[
\text{Optimal } d_{BAL} \equiv 41/0.0029 \equiv 17d 
\]

(19)

But we must not forget that: [1] The two errors (eqs.13&18) are of quite different type. [2] An ancient scientist would instinctively sense eq.18. [3] There’s no evidence that any ancient (or modern?) knew of eq.13; if he had, he’d have compensated, either by correcting for it (thus positive errors in Table 3?) or suppressing its effect via modest-sized \( d \).
K Hipparchos’ −157/6/28 Dawn Summer Solstice

[Thanks to DIO refereeing, albeit (uncharacteristically) late in this case, §§K-P have been rethought, recalculated, & rewritten (2018 Winter): prior mistakes fixed & new finds added.]

K1 Hipparchos c. −157 was using past records of eclipse-times to start building his famous 600° eclipse canon (§M2; Rawlins 1991W §M7), a list which included Hipparchos-computed solar longitudes \( \phi \) for each eclipse’s historically known time. Later, these \( \phi \) were brought in when he analysed eclipse-trios. In Rawlins 1991W §K9, we found that his math-analysis of eclipse-Trio B (§3 fn 5) used \( \phi \) computed (for each eclipse-time) from what we dubbed his “EH Orbit” (founded −157), which was afflicted with terrible apogee \( A = 44° \) & eccentricity \( e = 3° \) by, taking (Rawlins 1991W §K8) EH’s S.Solst — via indoor math — from Kallippos’ −329/6/28-epoch calendar (accumulated error +4°3 3 in the 179° interim), due to its over-long-year length, \( Y_c \) (§C1). From §C1 & eq.1 (see Tables 2&3):

\[
EH \text{ Summer Solst} = -329/6/28\ 1/4 + 172 \cdot 365\frac{3}{4}/14 = -157/6/28\ 1/4
\]

K2 Relative to the present analysis, the key point to notice is this: Hipparchos in −157 would not have computed a solstice from a predecessor’s calendar unless he didn’t yet know how to observe a solstice reliably. (The poetic irony here is that before his career was done, Hipparchos left us §§J5) THE most accurate outdoor-observed solstice that survives from antiquity, the error in which is merely about an hour. See Table 3.)

K3 If the young Hipparchos needed to resort to an earlier astronomer’s calendar to obtain his −157/6/28 dawn solstice (eq.22, used for constructing his EH solar orbit of −157: §K1 or Rawlins 1991W §§K8-K9), then where did his newly discovered (§A) −157/6/26 solstice come from? And when? Rigid impediment to casual thinking hereabouts: his calculation uses of the EH orbit’s tables as late as −145 (Rawlins 1991W §§M4-M6) shows that the 6/26 replacement-improvement was not adopted immediately.

K4 Duke’s idea (people.sc.fsu.edu/~dduke/Duke-Neugebauer-2.pdf) that the −157/6/26 solstice was empirically determined to have occurred at 21°, seems to be based upon his perceptive recognition of Meton’s fingernail: multiplying the papyrus’ tropical yearlength (convincingly extracted from it by Tihon 2010 p.5)

\[
Y_p = 365\frac{1}{4}/1 - 1^2/300 = 365.24676
\]

(23) times the 274° gap since Meton, and adding the product to Meton’s S.Solst, as misunderstood by Hipparchos & Ptolemy (§C1), produces:

\[
S.Solst = -431/6/27\ 1/4 + 274 \cdot Y_p = -157/6/26\ 20^3/43° = -157/6/26\ 21^4 (24)
\]

However: [a] All now known Hipparchos cardinal point data are rounded to the quarter-day. [b] In reverse, eq.24’s 21° time produces yearlength about 365\(\frac{1}{4}\)3° − 1°313, not 1°309, and so doesn’t solve eq.23’s origin. [c] In −157, Hipparchos wasn’t yet (§L4) sky-observing at a level likely to find an accurate solstice such as that proposed. [d] The papyrus says that the −157/6/26 solstice occurred at an unknown number of hours of the day not night.

K5 Potential resolution of [a]-[d]: if the papyrus said “12 hours of the day” (18° or 6 PM), that would make the gap from Meton (−431/6/27 1/4) to Hipparchos (−157/6/26 3/4) equal to 100077°1/2. But the ancient scholar who created eq.23 could have accounted for seasonal hours’ solstitial day-lengthening, taking 14°3/4 as the nearest klimate (of Almajest 2.6’s traditions) to a mean between Athens & Nicaea’s GD Book 8 longest days (Diller 1984):

\[
Y_p = (100077° + 14°3/4)/274 \pm 365\frac{1}{4}/1 - 1°309
\]

(25) (We here assume early Hipparchos didn’t know of or ignored small longitude differences.) Had the 14°3/8 Athens klimate (GD 3.15.22) been used, the remainder would’ve been −1°308°, perhaps an alternate value, as suggested by the P.Fouad 267A left column’s remainder +3°308° (Tihon 2010 p.7). Either way Fouad Hipparchan precession appears (but note fn 16) exactly or nearly 4°308° = 1°777°777° (vs actual 1°772° then, hinted at Almajest 7.2 (“not less”) but not explicitly relayed there (& a better figure than Ptolemy’s 1°100°). [Or reverse? Prior 1°77° precession estimate times 4→308°?]

L When Was the −157/6/26 3/4 Solstice Observed?

L1 The seemingly odd title of this section is not meant facetiously. (Though it puts one in mind of humor at the level of what-was-the-color-of-George-Washington’s-white-horse?) It is deliberate — because we are faced with a weird contradiction, two different dates for the same event, the −157 S.Solst: −157/6/28 1/4 (§L2) vs −157/6/26 3/4 (papyrus: §§K5).

L2 Rawlins 1991W⁹ (see §K1 above) has shown that Hipparchos’ eclipse-trios A&B cannot closely enough fit Almajest 4.1’s intervals for a solar eccentricity less than 3°. (And

⁹ Parts of Rawlins 1991W are written in an anti-tyrannical spirit which is bound to offend anyone unfamiliar with the cult that has for decades financially puppetized most of the history of ancient astronomy community, to its tragic cost in competence, refereeing, neutrality, and most importantly: valid history. If the History of Science Society can (fn 10) wine and stomach DR’s idiosyncratic writing style [once upon a time] — see Rawlins 2018A, in order to get at the truth of Hipparchos’ early observations & lunisolar elements, then fair-minded individual investigators ought to be able to manage same — for P.Fouad 267A will never be understood if §§K-O are discounted.
Trio B can’t fit an apogee A above 50º. Both limits are grossly discrepant vs the standard $\text{Almagest PH}$ orbit’s e = 2º 1/2. A. 65° + 1/2). Rawlins 1991W found that this clash is neatly accounted for by a huge error (over 1º) in $\text{solstice}$ — and that the EH orbit satisfying this glaring oddity is also consistent (like no later A.Excel) with the quite erroneous 157 Autumn Equinox (off by 1º, nearly half a day) reported at $\text{Almagest 3.1}$ and is consistent with a 157 solstice at 6/28 1/4 (§L1; Table 2), exactly where the Kallicippe calendar has it. EH was used by Hipparchos until his adoption of the later-canonical PH orbit in — 145, when EH’s role in Trio A’s Frankenstein-orbit solution proves Hipparchos anchored at S.Solst — 157/6/28 1/4 right up until — 145, not at the Fouda papyrus — 157/6/26 3/4. L3 L.I. 1/2, A. = $\text{e}$ = 10 $\text{3.1}$ contradicts it. Thus the EH orbit giving way to PH — as we see from multiple coherent indici: §L3, consecutive-triplet orbit-base (fn 13), — 145 V.Excel’s capper PH-rôle (idem), and Physkon’s — 145 accession (§03). Extra hint: eq.26’s ultra(extraordinarily?) neatness. (Also: — 145 minus — 157 = 12º 0” mod 4º). L5 L. from the — 146/6/26 1/2 S.Solst, Hipparchos need only go back 11 Kallicippe years, to create the “observed” — 157/6/26 3/4 S.Solst of his P.Fouda 267A tables & could’ve even more easily extrapolated 1º ahead to ensure a — 146/6/26 3/4 S.Solst (if not confirmed by year-later outdoor re-observation) for establishing his ultimately canonical-regnal — 145 PH orbit (for §L4). Pseudo-observed solstice-hours Kallicippe extrapolated from his — 146 solstice-hour for — 157 & — 145 would differ acceptably little from extrapolations based on Hipparchos’ yearlength (eq.23 or eq.32): 53 & 5º, resp. L6 L. as seen at Rawlins 1991W §§K4&k, Hipparchos was in — 157 searching for a S.Solst not by outdoor observation but by indoor calculation. Which tells us that he at this time didn’t know how to measure a solstice, nor even how to choose an expert who did. Perhaps it was just convenient (§3 fn 6) to stick with the increasingly inaccurate Kallicippe calendar, revered as that (too)long-standard among astroglogers, most of whom ignore the outdoor sky, Hipparchos later becoming the 1º known major exception. This discussion occasions our tabulation of the indoor solstices we have from antiquity (Table 2, chronologically ordered according to date of creation), including Ptolemy’s well-known 140 AD fraud at $\text{Almagest 3.1}$. The papyrus’ Hipparchos solstice (2º in Table 2) is only technically an indoor observation, as noted at fn 17: the accuracy of its outdoor procurator, the — 146/6/26 1/2 S.Solstice (§M4), transferred faithfully (§L5) to the — 157/6/26 3/4 extrapolation. M Solving the — 157 Double-Solstice Mystery M1 M. The ultimate implication of the foregoing is weird but simultaneously satisfies the various above-mentioned evidential features: following Hipparchos’ outdoor capture of the — 146/6/26 1/2 solstice, the papyrus” — 157/6/26 3/4 solst was extrapolated from it could allow an accurate solstice (§G1), but their sheer size (half a day!), and the proximity of their mean (12º) to the 16º error characteristic of an asymmetric gnomon, suggest sufficient crudity as to cast doubt (independent of §L3) on whether he got an accurate outdoor S.Solstice ere Rhodes-arrival. 13 No Hipparchos orbit until PH gives with Fouda’s — 157/6/26 3/4 S.Solst. But the PH orbit could not exist until the — 145 V.Excel. An orbit’s 3 required empirical cardinal-pt bases were best arranged consecutively, and no Hipparchos Winter Solst was used for orbits. (Just for finding obliquity & latitude, as also 100º earlier. Rawlins 1982G) So the V.Excel’s S.Solst-A.Excel triplet producing the final PH orbit used — 146/6/26 3/4. 14 Tiتون 2010 p.7 proves col.3 adopted — 1º/309. Col.3’s — 657-epoch table was completed (— 145) before computation of his then-still-incompletely-calculated eventual PH F-table, which (eq.32) rounded to — 150000 and used $\text{e}$ exactly fitting Alm 3.2 (§N1 item 5) [via the same PH yearlength. Did young Hipparchos use (§M2) epoch Phil 1 (— 32) for astronomy while adopting epoch — 657 for his mathematical study? — only later finally expanding back $\text{c.600}$ from his time to Nab 1 for PH’s F-table, which effectively went back c.1200$^0$ to c.1350 for early eclipses: www.dioi.org/thr.htm#bkv.
simply by subtracting 11° of motion. (Thus replacing the awful EH solst., −157/6/28 1/4, in future editions of his horoscopic publications, such as the material used by the P.Fouad 267A astrologer.) Moreover, Tihon 2010 (p.2) found that (along with parallel columns for sidereal & "tropical" longitudes) the papyrus' ephemeris retains a Kallichippically-computed column of solar longitudes (at quarter-century intervals) — startlingly consistent with Rawlins 1991W §K’s proposals that [i] EH’s mean solar motion was Kallippic and [ii] EH’s foundation S.Solst. was −157. Tihon discovered from the papyrus that its practical epoch15 was −657/2/4 (Nab 90 Thoth 1), running 500° (Egyptian years of 365 1/4) each and (like Pooley’s Handy Tables & Almajest 6.3) at 25° per line. Given the Fouad-astrologer's addition of mean solar motion for 21h to an integral number of days from epoch, we know (since his horoscope is for 3 AM) his −657/2/4 epoch was 6°.

M2 From these findings & his ultimate immortality (& Fouad’s citing “nativity” as its calculational purpose), we can guess Hipparchos had published an internationally popular, profitably-multiple-tradition astrological manual in −157, including a purely Kallippic table for mean solar longitude, eventually going 500° into the past and perhaps 100° more into the future: 600° in all, possibly vs §K1 the basis for Pliny 2.8.53’s reference to Hipparchos’ 600° of calculations. The curious failure of the papyrus’ (pre-Almajest) astrologer to cite any work later than −157 may indicate that Hipparchos’ mature researches were more scientifically than popular and were primarily intended for an astronomical not astrological audience. (Financed by selling horoscopes & manuals for? And-or govt’s support?) When in −146 he realized how wrong the EH orbit’s solstice was, he appended at least the column of mean solar longitudes based upon Metonic YP (eq.23). We may compute the Kallippic column’s ε0 by working backwards from Hipparchos’ epoch (eq.1), when true solar longitude \( \phi = 90° - \) −329/6/28 1/4, which (by PH’s c&a) is when mean solar longitude \( f = 90°59' \). Result: the papyrus’ middle (Kallichippic) column’s mean-longitude-at-eclipse for −657/2/4 6° ([M1] was solar ε0 = 309°03' (vs actually 306° 7').

M3 Fouad bears 3 columns of computed φ: [1] left,16 [2] Kallippic or “mean” (middle). [3] Metonic “tropical” (right). The last is PH (but for eq.23’s yearlength): we revolve back (again from \( f = 90°59' \)) for the 182767/2/12 from −157/6/26 3/4 to −657/2/4 1/4, finding ε0 = 308°56' (49° for 14°3/4 klima). (This & §M2 rounded to ε0 = 309° for computing?)

M4 Finally, in answer to this section’s semi-facetious titular question: the −157/6/26 solstice was truly17 outdoor-observed by Hipparchos at 146/6/26 1/2 & then — to replace his erroneous indoor epoch −157/6/28 1/4 solstice — he Kallichippically-reconstituted18 it back at −157/6/26 3/4, with but tiny concomitant error ([5.5, Table 2] as he was fully aware. So, was Pliny 2.5.27 wrong in claiming that not even god can change the past?

N Statistical Impregnability of the −157/6/28 1/4 Solstice’s Adoption

N1 To understand what DIO has accomplished here regarding Hipparchos’s solar theory, let us catalog the FIVE types of fits simultaneously achieved at Rawlins 1991W §§K&M:

15 Almajest 3.1 shows that Hipparchos’ solar observations were dated according to the number of years after “the death of Alexander” or equivalently epoch Phil 1, the ascension of Philip III: §P3.13.
16 The 365+4/4(102 2/3) yearlength was far closer to the real anomalistic year (remainder: +14°102') than sidereal. Left-column yearlength is consistent with remainder −7°1/2 in ancients’ key 345° equation (Rawlins 1996C §C: implicit yearlength = 365°13/100, used in Almajest 4.2 to find 11° 3/4. Fouad does not explicitly whether left-column is for sidereal (Tihon 2010 p.6) or apsidal precession.
17 The −157/6/26 3/4 solstice is not at all a fabrication. Hipparchos knew that extrapolating the −146/6/26 1/2 solstice would it produce a result differing but ordmag 1° from the truth if his −146 observation was accurate. Hipparchos no more thought of extrapolating-reconstituting it as dishonest than he thought it a trick to find a solst. by eq.5’s interpolation. Neither resulting datum is a direct observation, but the procedure is scientifically proper and justifiable in both cases.
18 One may hypothesize the reverse: indoor −146 solstice reconstituted from outdoor −157 solstice. But aside from the question ([L4] of Hipparchos’ crude instruments in −157, was getting-rich (fn 20) why he waited 12° before adopting (fn 13) the −145/6/26 18° S.Solstice to found his PH tables?

19 See www.ioi.org/jha.htm#pdf. JHA Pb (!) paper Duke 2008W displays not only the mistake just analysed but 3 others equally obvious to nonzombie refereeing: 2 at §B4 & 1 more at fn.10’s final line.
O Recovering Hipparchos’ Lost – 146 Solstice

O1 From –145 to –134, Hipparchos’ mainstay solar orbit was PH, later appropriated (essentially unaltered) by Ptolemy (Almajest 3.2&6), standard among astrologers for centuries, cited as “perfect” by Julian the Apostate (1:429). 5000 later, through by then differing from reality by 2° or 2′! Rawlins 1991H §5-D showed that in –134 Hipparchos abandoned PH and adopted the Metonic orbit. But the question that has never previously been answered (even asked) is: whence came the S. Solst needed for the PH orbit?

O2 The only 2 years Almajest 3.1’s Hipparchos cardinal-point data lists both equinoxes: –146 AE to –145 AE & –142 VE to –141 VE, the latter barred by its A.Eq’s discord with the PH orbit. His 1st outdoor-observed S.Solst cannot be –145/6/26 3/4 since Hipparchos used (Rawlins 2009E §B5) the PH orbit months earlier to place the mid-eclipsed Moon (–145/4/21), so the –146/6/26 1/2 S.Solst was his 1st Rhodos sky-record.

O3 Almajest 3.1’s collection of Hipparchan cardinal-point observations cites only Autumn20 Eqinoxes before his –145/3/24 V.Eq captured capture (§O2) of his 3rd Rhodos solar-cardinal pt data, of the three needed to compute (like Neugebauer 1975 pp.58-60) his PH orbit, just in time to figure mid-eclipse for his –145/4/21 measure of Spica’s place (Almajest 3.1). [Added 2018/2/10. The timing suggests: did he move to Rhodos for its good weather just before the –146 S.Solst, partly to ensure that he wouldn’t be ‘miss measuring the –145 eclipse’? Almajest’s PH orbit (epoch Phil 1 Thoth 1 = 323/11/12 Alex App Noon; elements at Rawlins 1991 §K10) gives solar true longitude $\phi_{AE}$ for his –145/9/27 1/4 A.Eq, only 2°1/4 before the regnal epoch Ptolemy Physkon 1 Thoth (Toomer 1984 pp.11&133), with PH mean anomaly $g = 116^\circ 2/3$ (Almajest 3.7; Neugebauer 1975 p.59;)

$$\phi_{AE} = 227^\circ 2/3 + \frac{360^\circ \cdot 6967^\circ 3/4}{Y_H} - \arctan \frac{\sin 116^\circ 2/3}{24 + \cos 116^\circ 2/3} = 180^\circ 00'00''$$ (26)

It appears21 that the PH solar mean-longitude-at-epoch (same as Ptolemy’s at Almajest 3.2) $\epsilon_o = 227^\circ 2/3$ was set by Hipparchos to ensure the exactitude of eq.26, consistent with the PH orbit’s launch upon –145 V.Eq’s capture. So we have traced the A.Eq-origin of Ptolemy’s hitherto-unexplained Nabonassar 1 Thoth $\epsilon_o = 330^\circ 45'$ (Almajest 3.2&7), 424th prior to Hipparchos’ Phil 1 epoch. Hipparchos thus gave calendrical priority to the A.Eq’s (fn 20). Anyway, it’s obvious that –146 SS to –145 AE (§O2) was the period of the

P PH Yearlength’s Origin? Hipparchos’ Ingenious Great-Year Cycle

P1 From his –145 V.Eq, S.Solst, & A.Eq, Hipparchos computed (method: Almajest 3.4; Rawlins 1991H §C3) three of his final PH orbit’s elements: $\epsilon_o, \epsilon, & A$. But the 4th of the required 4 elements, the mean motion $F$, must depend in part upon earlier astronomers’ observations. Except for hist.astron’s dearest archons, scientific historians know how ancients estimated year-lengths: by comparing solstices centuries apart.

P2 In order to gauge ancient solstices’ and year-lengths’ accuracies, we need to know the actual values at that time. For Hipparchos’ era, the true mean tropical22 year was (Rawlins 1999 §C10) about 365.2425.

Actual Hipparchos-Era Tropical Year-Length $\pm 365.2425 - 365.2415 = 365.2430$ (28)

The foregoing rounding happens to be equal to Jesuit Christopher Clavius’s Gregorian-rule year-length, established 17 centuries later (when the year was nearer 365.2423), and which we live-by today. (See puzzle at DIO 4.2 p.2, instantly solved by K.Pickering & R.Freitag.)

P3 And there is an extra factor which is oft-forgotten, namely (in 23): each of the four cardinal-points has its own year-length — generally differing from the others by a few ten-thousandths of a day. Both their relative proportions and their absolute lengths vary secularly. For the present discussion, we should know the Hipparchos-era S.Solst value:

Actual Hipparchos-Era S.Solst Year-Length $= 365.2421$ (29)

To measure empirical error, compare ancient figures to eq.29; to measure vs mean year-length (which ancients thought they were determining), compare to eq.28.

P4 As soon as his –146 S.Solst measurement was in hand, Hipparchos returned to his earlier babbling with Meton, which had led (§K5) to a yearlength tantalizingly close to compatibility with Meton’s definition from his ratio (still used for modern Easter):

Meteoric Year $Y_M = 235$ months/19

which (via $\Box4$ eq.4) requires

$$Y_{HM} = (235/19) \cdot 29231500800020'' = 365^\circ 1/4 = 1^\circ 00''$$ (31)

20 Or did Hipparchos have an unusual calendrical interest in the Autumn Equinox, since it was near the Egyptian calendar’s start (Thoth 1) in his era? During the year of his UH-founding –134 S.Solst (eq.4), his –134/9/24 A.Eq occurred smack-on Thoth 1 (Rawlins 1991H fn 14); and his UH solar mean longitude 180° occurred at 10° on Thoth during the UH orbit’s –1279/24/24-hour epoch (ibid eq.28). During the 11° gap ‘twixt Hipparchos’ –157 & –146 observations, did astrological tables’ (Tihon 2010) sales make him rich enough to return to creativity (Rachmaninov [www.dioi.org/rar.htm] parallel 1917-1926)? — moving to clear-skied Rhodos, to facilitate fulfilling a dream of founding astronomy empirically.

21 Note: –145 S.Solst proposed for ultimate PH orbit on thin evidence as early as Rawlins 1985H. And see Rawlins 1991W §M6, where it also noted that –145 was a ‘regnal year’. Ptolemy VII Physkon’s. See Rawlins 1991H fn 7 for Physkon 1 Thoth 1, which usefully clinches –145 as PH’s epoch, crucially since eq.25 adjusted for other nearby years would be nearly as well-fitting for A.Eq; –145’s V.Eq plus $+7.9^\circ$ S.OFF & S.Solst off $+0.7^\circ$. (Rawlins 1991W §M4’s best Frankiestein-orbit fit was for $A = 65^\circ$, but $A = 65^\circ 1/2$ fits Trio A’s data nearly as well & it’s the Hipparchos apogee preserved even centuries later at Almajest 3.7; Neugebauer 1975 pp.58f. The superb analysis of van Dalen 1994 showed that the Almajest 3.6 anomaly table’s numbers were actually generated from $A = 66^\circ$.)

22 To consider an extreme case: if a S.Solst that occurred at 14:00 were measured by the observer as having occurred at 15:01, which he accurately rounded to traditional 1/4 precision (i.e., to 18″), an O–C error of merely $18''$ would effectively quadruple, appearing to us to be a 4° O–C error. See fn 8.

23 Technically, what has long been called a “tropical year” is a misnomer, since it refers to the sidereal year minus the effect of precession. But that standard figure — eq.28 — was not ($\Box3$) the same as either of the two solstitial years: i.e., the mean Sun’s returns to the Summer Tropic & Winter Tropic. Nor the same as the years measuring the mean Sun’s returns to the Vernal & Autumnal Equinoxes. (You’ll have to ask the esteemed Journal for the History of Astronomy about the Winter Equinoctial Year: §B1.) Note that in antiquity the average of the years of the S.Solst&W.Solst virtually equaled eq.28, as did the average of the V.Eq&A.Eq years.
Comparing his \(-146/6/26\) 1/2 solst with his hugely erroneous \(-431/6/27\) 1/4 dawn-version of Meton’s solst (\(-1^1/2\) off, eq.4), 104095\(^1/4\) 1/4 earlier, he found (for best Almajest 2.6 Athens-Rhodos klima \(1^{4/1}/2\) PH-vs-Meton remainder = \(-1^7/300.66\); trivially rounding:
\[
Y_{p-M} = \frac{104095}{431}{}^1/4 + \frac{1}{2} = 365^1/4 - 1^4/4 \approx 365^1/4 - 1^4/400 \approx 365^1/4 - 24667 = Y_H (32)
\]
This is the 1\(^{st}\) time a modern has empirically justified by calculation astronomer Hipparchos’ famous yearlength \(Y_H\), adopted by Ptolemy and used for centuries thereafter. From here, Hipparchos devised his astonishing Great Year vision with its 5-stages geometrically embedded integral-return cycles (304\(^1/4\), 608\(^1/2\), 1217\(^1/4\), 2434\(^3/4\), 4868\(^2/4\)), fully unfurled at Rawlins 2002A fn 14, 16, 17. This Great Year fixed his long-view yearlength \(Y_G\):
\[
Y_G = 365^1/4 - \frac{1}{4} \approx 1^4/400 \approx Y_K - \frac{1}{4} \approx 365^1/4 - 24671 (33)
\]
We note that period 304\(^1/4\) (which is exactly 4 Kallicptic cycles and 16 Metonic cycles) is clearly attested for Hipparchos by Censorinus (Heath 1913 p297); for 4868\(^2/4\), see fn 10 [4].

But then, 12\(^{th}\) later, along came Hipparchos’ – 134/6/26 1/4 S.Solst, 5\(^{st}\) earlier than predicted by the PH orbit. (For potential effect, compare eq.35 to eq.32.) So did Hipparchos switch to a new year-length value? No — he instead (like the conservatism of Meton’s solst (eq.30), while not close to the mark, is the best of a
\[
\frac{24667}{24671} = 1 - \frac{1}{300.66} = \frac{24667}{24671} = Y_H (32)
\]
Hipparchos’ Indoor/Outdoor Solstices 2012 Rev 2015&2018 DIO 20 ‡2

Hipparchos’ Indoor/Outdoor Solstices 2012 Rev 2015&2018 DIO 20 ‡2

Q Preconception’s Wages: Hipparchos Neglects Kallippos’ Solstice

The contention of Rawlins 1999 [§D4] that the tropical year-length estimates we have from antiquity (with the exception of eq.35) flock quite unrandomly around the artificial Metonic value of eq.31. These results vindicate Tobias Mayer’s solution (modernly rediscovered by R.Mercier, K.Moesgaard, N.Swerdlow, & DR.) of the source of the systematic error in the Hipparchos-Ptolemy solar tables, namely, the Hipparchos year mimicked the Metonic lunis-solar year-length: eq.31. So preconception from (evidently) near-universal belief in eq.31 caused Hipparchos to miss the opportunity to acquire the 1\(^{st}\) accurate tropical year-length. Survey his career-long search for a trustworthy ancient-to-him solstitial anchor: [a] In – 157, he uses Kallippos’ – 329 Summer Solst to anchor EH. [b] While 12\(^{th}\) later adopting Meton’s eq.30, he observes the – 145 S.Solst but finds it won’t work Metonically (eq.31) with Meton’s own – 431 S.Solst unless (§C1) Meton’s “start of day” is (false): eq.1 taken to mean dawn, thus – 12\(^{nd}\) – fudging – 431 S.Solst and his own PH. [c] When, 11\(^{th}\) later, his new – 135 S.Solst observation jars vis-à-vis the previous – 145 one, he shifts anchor from Meton’s – 431 solst to Aristarchos’ – 279 solst for UH, in order to maintain (§Q1) his year-remainder at c. – 1\(^{st}\)300. [d] But for his ultimate anchor, Hipparchos never goes back to the only accurate solst of the now-known lot: Kallippos’ where he started (item [a] above; or §K1). This takes us into the plainest proof of Metonic preconception’s grip (§D6&8), & an obvious, previously-unasked question: why did Hipparchos never compare either of his outdoor solstices to Kallippos’, whose S.Solst offered longer baselines than Aristarchos’. Had he done so for his 1\(^{st}\) empirical solstice (– 146), he’d have found (interval 183\(^{d}\)), treating seasonal hours naïvely:
\[
Y_{H1-K} = \frac{66839}{329} + \frac{1}{4} + 365^1/4 - 1^4/4 \approx \frac{66839}{329} + 365^1/4 - 1^4/4 = 365^1/4 - 1^4/122 \approx 365^1/4 - 1^4/122 \approx 365^1/4 - 1^4/122 \approx 365^1/4 - 1^4/122 \approx 365^1/4 - 1^4/122 (36)
\]
and, for his 2\(^{nd}\) empirical solstice (– 134), using an interval of 195\(^{d}\):
\[
Y_{H2-K} = \frac{71229}{329} + \frac{1}{4} = \frac{71229}{329} - 1^4/780 \approx 365^1/4 - 1^4/111 (37)
\]

These 2 potential (historically-unrealized) year-lengths’ errors vs the real mean year (eq.28):
\[
Y_{H1-K} = 1^0/0, Y_{H2-K} = 2^0/1. And vs S.Solst yr (eq.29): Y_{H1-K} = 0^0/0, Y_{H2-K} = 0^0/0 (38)
\]

Q2 Despite solstices’ failure to yield an accurate tropical year (due to truncations, prejudice for eq.31, & not choosing Kallippos’ solstice as earlier anchor), solstices nonetheless contributed to gradual improvement of the solar orbit, being (§G1) the most reliable of the 4 cardinal points. Whatever the quality of the calendrical uses made of them, the 4 recoverable outdoor ancient solstices (Table 3) were so conscientiously accomplished by the methods we discussed at the outset (culminating in §J), that all four are accurate within their quarter-day rounding — rounding which (§O4) has made it impossible to tell whether the pre-rounded values were more than trivially in error. As noted at §O5, this is yet another vindication for the high level of ancient Greek science, and for those who’ve defended it.
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